TEST 2

(Math 140-C)

Solve the following problems. Show all your work in the space under each problem.

1. (a) **True** or **False**: The vector $\vec{u} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$ is an eigenvector of $A = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$. (15 pts)

   (b) Find the eigenvalues of $A = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$.

   (c) Find the corresponding eigenvectors for each eigenvalue of A.

2. Given that $\vec{u} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$, then: (15 pts)

   (a) Find the cross product of $\vec{u}$ and $\vec{v}$.

   (b) **True** or **False**: $\vec{u}$ and $\vec{v}$ are parallel.

   (c) Find the area of the parallelogram generated by $\vec{u}$ and $\vec{v}$.

3. Given the plane *M*: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + s\begin{pmatrix} 2 \\ 7 \\ 1 \end{pmatrix}$ and the line *l*: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} + t\begin{pmatrix} -2 \\ -6 \\ -2 \end{pmatrix}$:

(a) Express the plane *M* in the form $ax + by + cz = d$. (10 pts)

(b) Find the point of intersection of the plane *M* and the line *l*.

4. Given $A = \begin{pmatrix} 2 & 0 & 7 \\ 0 & -1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$, find : (15 pts)

(a) The det(*A*).

(b) Find the cofactor matrix of *A*.

(c) Find A^{-1} using the adjoint of *A*.

5. Let $T : R^3 \to R^3$ be the reflection about the plane $x + z = 0$. Find: (20 pts)

(a) The matrix *A* that describes the transformation *T*.

(b) The *kernel* of the transformation *T*.

(c) All the *fixed points* under T.

(d) The image of the plane $x - 2y + 3z = 1$ under T.

6. (a) Find the 3×3 matrix that rotates a vector by $30°$ around the z-axis, and then projects it on the xy-plane. (10 pts)

(b) Use the matrix above, to find the image of the vector $\vec{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

7. Consider the triangle with vertices $A = (0,1,0)$, $B = (0,3,0)$, and $C = (0,2,2)$.

(a) Find the 4×4 matrix that first rotates the triangle ABC by $90°$ about the z-axis, and then translates it by (-4,0,0) .

(b) Use the above matrix to find the coordinates of the shifted triangle.

(c) Make all relevant graphs that show the original triangle, the rotation, and the shifted triangle. (15 pts)

Some useful formulas:

(1) **Reflection** about the plane $a\,x + b\,y + c\,z = 0$:

$$A = \frac{1}{a^2+b^2+c^2}\begin{pmatrix} -a^2+b^2+c^2 & -2ab & -2ac \\ -2ab & a^2-b^2+c^2 & -2bc \\ -2ac & -2bc & a^2+b^2-c^2 \end{pmatrix}$$

(2) **Projection** onto the plane $a\,x + b\,y + c\,z = 0$:

$$A = \frac{1}{a^2+b^2+c^2}\begin{pmatrix} b^2+c^2 & -ab & -ac \\ -ab & a^2+c^2 & -bc \\ -ac & -bc & a^2+b^2 \end{pmatrix}$$

(3) **Rotation** about the unit vector $\vec{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ by an angle θ :

$$A = \begin{pmatrix} a^2(1-\cos\theta)+\cos\theta & ab(1-\cos\theta)-c\sin\theta & ac(1-\cos\theta)+b\sin\theta \\ ab(1-\cos\theta)+c\sin\theta & b^2(1-\cos\theta)+\cos\theta & bc(1-\cos\theta)-a\sin\theta \\ ac(1-\cos\theta)-b\sin\theta & bc(1-\cos\theta)+a\sin\theta & c^2(1-\cos\theta)+\cos\theta \end{pmatrix}$$